J Stat Phys (2008) 131: 717–731
DOI 10.1007/s10955-008-9481-7

Is Brownian Motion Sensitive to Geometry Fluctuations?

C. Chevalier · F. Debbasch

Received: 26 June 2007 / Accepted: 4 January 2008 / Published online: 14 March 2008

Abstract Many situations of physical and biological interest involve diffusions on manifolds. It is usually assumed that irregularities in the geometry of these manifolds do not influence diffusions. The validity of this assumption is put to test by studying Brownian motions on nearly flat 2D surfaces. It is found by perturbative calculations that irregularities in the geometry have a cumulative and drastic influence on diffusions, and that this influence typically grows exponentially with time. The corresponding characteristic times are computed and discussed.

Keywords Brownian motion · Stochastic processes on manifolds · Lateral diffusions

1 Introduction

Stochastic process theory is one of the most popular tools used in modelling time-asymmetric phenomena, with applications as diverse as economics ([21, 22]), traffic management ([15, 20]), biology ([2, 8, 10, 16]), physics ([23]) and cosmology ([5]). Many diffusions of biological interest, for example the lateral diffusions ([4, 17]), can be modelled by stochastic processes defined on differential manifolds ([9, 12, 13, 18]). In practice, the geometry of the manifold is never known with infinite precision, and it is common to ascribe to the manifold an approximate, mean geometry and to assume irregularities in the geometry have, in the mean, a negligible influence on diffusion phenomena ([1, 3, 4, 6, 19]). The aim of this article is to investigate if this last assumption is indeed warranted.

To this end, we fix a base manifold $\mathcal{M}$ and focus on Brownian motion. We introduce two metrics on $\mathcal{M}$. The first one, g, represents the real, irregular geometry of the manifold; what an observer would consider as the approximate, mean geometry is represented by another

C. Chevalier (✉) · F. Debbasch
Université Pierre et Marie Curie-Paris 6, UMR 8112, ERGA-LERMA, 3 rue Galilée, 94200 Ivry, France
e-mail: chevalier_claire@yahoo.fr

F. Debbasch
e-mail: fabrice.debbasch@gmail.com

metric, which we call $\bar{g}$; to keep the discussion as general as possible, both metrics are allowed to depend on time.

We compare the Brownian motions in the approximate metric $\bar{g}$ to those in the real, irregular metric g by comparing their respective densities with respect to a reference volume measure, conveniently chosen as the volume measure associated to $\bar{g}$. Explicit computations are presented for diffusions on nearly flat 2D surfaces whose geometry fluctuates on spatial scales much smaller than the scales on which these diffusions are observed. We investigate in particular if the densities generated by Brownian motions in the real, irregular metric g coincide on large scales with the densities generated by Brownian motions in the approximate metric $\bar{g}$. We perform a perturbative calculation and find that, generically, these densities differ, even on large scales, and that the relative differences of their spatial Fourier components grow exponentially in time; on a given surface, the characteristic time τ at which the perturbative terms become comparable (in magnitude) to the zeroth order terms depends on the amplitude ε of the irregularities and on the large scale wave vector k at which diffusions are observed; we find that τ generally scales as $-(\nu^{-2}\ln(\varepsilon/\nu^{1/2})) \times (1/|K^*|^2\chi)$, where χ is the diffusion coefficient and $\nu = |k|/|K^*|$, K^* being a typical wave-vector characterizing the metric irregularities. Our general conclusion is that geometry fluctuations have a cumulative effect on Brownian motion and that their influence on diffusions cannot be neglected.

2 Brownian Motions on a Manifold

2.1 Brownian Motion in a Time-Independent Metric

Let $\mathcal{M}$ be a fixed real base manifold of dimension d. Let g be a time-independent metric on $\mathcal{M}$. This metric endows $\mathcal{M}$ with a natural volume measure which will be denoted hereafter by $d\mathrm{Vol}_g$. If $\mathcal{C}$ is a chart on $\mathcal{M}$ with coordinates $x = (x^i)$, $i = 1, \ldots, d$, integrating against $d\mathrm{Vol}_g$ comes down to integrating against $\sqrt{\det g_{ij}}\, d^d x$, where the g_{ij}'s are the components of g in the coordinate basis associated to $\mathcal{C}$.

There is a canonical definition of a Brownian motions on $\mathcal{M}$ equipped with metric g ([9, 11, 14, 18]). Quite intuitively, these Brownian motions are defined through the diffusion equation obeyed by their densities n with respect to $d\mathrm{Vol}_g$. Given an arbitrary positive diffusion constant χ, this equation reads:

$$\partial_t n = \chi \Delta_g n, \tag{1}$$

where Δ_g is the Laplace-Beltrami operator associated to g ([7]); given a chart $\mathcal{C}$ with coordinates x, one can write:

$$\Delta_g n = \frac{1}{\sqrt{\det g_{kl}}} \partial_i \left(\sqrt{\det g_{kl}}\, g^{ij} \partial_j n \right), \tag{2}$$

where ∂_i represents partial derivation with respect to x^i and the g^{ij}'s are the components of the inverse of g in the coordinate basis associated to $\mathcal{C}$. Observe that one of the reasons why this definition makes sense is that the diffusion equation (1) conserves the normalization of n with respect to $d\mathrm{Vol}_g$.

2.2 Brownian Motion in a Time-Dependent Metric

The preceding definition of Brownian motion cannot be used in this case because the diffusion equation (1) does not conserve the normalization of $n(t)$ with respect to the volume measure $\mathrm{dVol}_{g(t)}$ associated to a time-dependent metric. To proceed, we introduce an arbitrary, time-independent metric γ on $\mathcal{M}$, denote by $\mu_{g(t)|\gamma}$ the density of $\mathrm{dVol}_{g(t)}$ with respect to dVol_γ, and define the Brownian motion in the time-dependent metric $g(t)$ as the stochastic process whose density n with respect to $\mathrm{dVol}_{g(t)}$ obeys the following generalized diffusion equation:

$$\frac{1}{\mu_{g(t)|\gamma}} \partial_t \left(\mu_{g(t)|\gamma} n\right) = \chi \, \Delta_{g(t)} n. \tag{3}$$

Given an arbitrary coordinate system (x), (3) transcribes into:

$$\partial_t \left(\sqrt{\det g_{kl}}\, n\right) = \chi \, \partial_i \left(\sqrt{\det g_{kl}}\, g^{ij} \partial_j n\right), \tag{4}$$

which shows that the Brownian motion in $g(t)$ does not actually depend on γ. Moreover,

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{M}} \mathrm{dVol}_{g(t)} n &= \frac{d}{dt} \int_{\mathcal{M}} \mathrm{dVol}_\gamma \, \mu_{g(t)|\gamma} n \\
&= \int_{\mathcal{M}} \mathrm{dVol}_\gamma \, \partial_t \left(\mu_{g(t)|\gamma} n\right) \\
&= \chi \int_{\mathcal{M}} \mathrm{dVol}_\gamma \, \mu_{g(t)|\gamma} \, \Delta_{g(t)} n \\
&= \chi \int_{\mathcal{M}} \mathrm{dVol}_{g(t)} \, \Delta_{g(t)} n \\
&= 0.
\end{aligned} \tag{5}$$

Thus, contrary to (1), (3) conserves the normalization of $n(t)$.

3 How to Compare Brownian Motions in Different Metrics

Let $\mathcal{M}$ be a real differential manifold of dimension d. We first introduce on $\mathcal{M}$ a metric $\bar{g}(t)$ which describes what an observer would consider as the approximate, mean geometry of the manifold. The real, irregular geometry of $\mathcal{M}$ is described by a different metric $g(t)$. Consider an arbitrary point O in $\mathcal{M}$ and let B_t be the Brownian motion in $g(t)$ that starts at O. The density n of B_t with respect to $\mathrm{dVol}_{g(t)}$ obeys the diffusion equation:

$$\frac{1}{\mu_{g(t)|\gamma}} \partial_t \left(\mu_{g(t)|\gamma} n\right) = \chi \, \Delta_{g(t)} n. \tag{6}$$

We denote by $\bar{B}_t$ the Brownian motion in $\bar{g}(t)$ that starts at point O and by $\bar{n}$ its density with respect to $\mathrm{dVol}_{\bar{g}(t)}$; this density obeys:

$$\frac{1}{\mu_{\bar{g}(t)|\gamma}} \partial_t \left(\mu_{\bar{g}(t)|\gamma} \bar{n}\right) = \chi \, \Delta_{\bar{g}(t)} \bar{n}. \tag{7}$$

We will compare the two Brownian motions by comparing on large scales their respective densities with respect to a reference volume measure on $\mathcal{M}$. From an observational point of view, the best choice is clearly $\mathrm{dVol}_{\bar{g}(t)}$, the volume measure associated to the approximate, mean geometry of the manifold. The density N of B_t with respect to $\mathrm{dVol}_{\bar{g}(t)}$ is given in terms of n by:

$$N = \mu_{g(t)|\bar{g}(t)} n, \tag{8}$$

where $\mu_{g(t)|\bar{g}(t)}$ is the density of $\mathrm{dVol}_{g(t)}$ with respect to $\mathrm{dVol}_{\bar{g}(t)}$. The transport equation obeyed by N can be deduced from (6) and reads:

$$\frac{1}{\mu_{g(t)|\gamma}} \partial_t \left(\mu_{\bar{g}(t)|\gamma} N\right) = \chi \, \Delta_{g(t)} \left(\frac{1}{\mu_{g(t)|\bar{g}(t)}} N\right). \tag{9}$$

In a chart $\mathcal{C}$ with coordinates (x), (8) transcribes into:

$$N(t,x) = \frac{\sqrt{\det g_{ij}(t,x)}}{\sqrt{\det \bar{g}_{ij}(t,x)}} n(t,x) \tag{10}$$

and (9) becomes:

$$\partial_t \left(\sqrt{\det \bar{g}_{kl}(t,x)} N(t,x)\right) = \chi \, \partial_i \left(\sqrt{\det g_{kl}(t,x)} g^{ij}(t,x) \partial_j \frac{\sqrt{\det \bar{g}_{kl}(t,x)}}{\sqrt{\det g_{kl}(t,x)}} N(t,x)\right). \tag{11}$$

The precise question we investigate in this article is: how does the density N obeying (9) differ on large scales from the density $\bar{n}$ obeying (7)? Since this question is extremely difficult to solve in its full generality, we now concentrate on nearly flat 2D surfaces.

4 Brownian Motions on Nearly Flat 2D Surfaces

4.1 The Problem

We choose $\mathbb{R}^2$ as base manifold $\mathcal{M}$ and retain $\bar{g} = \eta$, the flat Euclidean metric on $\mathbb{R}^2$. The real, irregular metric of the manifold is still denoted by $g(t)$ and we define $h(t)$ by $g^{-1}(t) = \eta^{-1} + \varepsilon h(t)$, where ε is a small parameter (infinitesimal) tracing the nearly flat character of the surface. From now on, we will use the metric η (resp. the inverse of η) to lower (resp. raise) all indices.

Let us choose a chart $\mathcal{C}$ where $\eta_{ij} = \mathrm{diag}(1, 1)$. The tensor field $h(t)$ is then represented by its components $h^{ij}(t,x)$. A particularly simple but very illustrative form for these components is:

$$h^{ij}(t,x) = \sum_{nn'} h^{ij}_{nn'} \cos\left(\omega_{n'} t - k_n . x + \phi_{nn'}\right), \tag{12}$$

where $k_n . x = k_{n\,1} x^1 + k_{n\,2} x^2$ and both integer indices run through arbitrary finite sets. This choice has the double advantage of leading to conclusions which are sufficiently robust to remain qualitatively valid for all sorts of physically interesting perturbations h while making all technical aspects of the forthcoming computations and discussions as simple as possible. The *Ansatz* (12) will therefore be retained in the remainder of this article. Let us remark that perturbations $h(t)$ proportional to η amount to a simple modification of the conformal factor linking the 2D metric $g(t)$ to the flat metric η.

Equation (11) reads, in the chart $\mathcal{C}$:

$$\partial_t N = \chi \partial_i \left(\sqrt{\det g_{kl}(t,x)}\, g^{ij}(t,x) \partial_j \frac{N}{\sqrt{\det g_{kl}(t,x)}} \right) \tag{13}$$

or, alternately,

$$\partial_t N = \chi \partial_i \left(g^{ij}(t,x) \left(\partial_j N - N \partial_j l \right) \right), \tag{14}$$

where

$$l(t,x) = \ln \sqrt{\det g_{kl}(t,x)}. \tag{15}$$

4.2 General Perturbative Solution

The solution of (14) will be searched for as a perturbation series in the amplitude ε of the fluctuations:

$$N(t,x) = \sum_{m \in N} \varepsilon^m N_m(t,x). \tag{16}$$

Setting to 0 both coordinates of the point O where the diffusion starts from, we further impose, for all x, that $N_0(0,x) = \delta(x)$ and $N_m(0,x) = 0$ for all $m > 0$.

The function $l(t,x)$ can be expanded in ε, so that $l(t,x) = \sum_{m \in N} \varepsilon^m l_m(t,x)$ and one finds, for the first three contributions:

$$\begin{aligned} l_0(t,x) &= 0, \\ l_1(t,x) &= -\frac{1}{2}\, \eta_{ij} h^{ij}(t,x), \\ l_2(t,x) &= \frac{1}{4} \eta_{ik} \eta_{jl} h^{ij}(t,x) h^{kl}(t,x). \end{aligned} \tag{17}$$

Equation (14) can then be rewritten as the system

$$\partial_t N_m = \chi \Delta_\eta N_m + \chi S_m[h, N_r], \quad m \in \mathbb{N},\ r \in \mathbb{N}_{m-1}, \tag{18}$$

where the source term S_m is a functional of the fluctuation h and of the contributions to N of order strictly lower than m. In particular,

$$\begin{aligned} S_0 &= 0, \\ S_1 &= \partial_i \left(h^{ij} \partial_j N_0 + \frac{1}{2} N_0 \eta^{ij} \eta_{kl} \partial_j h^{kl} \right), \\ S_2 &= \partial_i \left(h^{ij} \partial_j N_1 + \frac{1}{2} \left(N_0 h^{ij} + N_1 \eta^{ij} \right) \eta_{kl} \partial_j h^{kl} - \frac{1}{4} N_0 \eta^{ij} \eta_{mk} \eta_{nl} \partial_j \left(h^{mn} h^{kl} \right) \right). \end{aligned} \tag{19}$$

Two remarks are now in order. Taken together, $S_0(t,x) = 0$ and $N_0(t,x) = \delta(x)$ imply that N_0 coincides with the Green function of the standard diffusion equation on the flat plane:

$$N_0(t,x) = \frac{1}{4\pi \chi t} \exp\left(-\frac{x^2}{4\chi t} \right). \tag{20}$$

Moreover, the fact that S_m is a divergence for all m implies that the normalizations of all N_m's are conserved in time. The initial condition $N_m(0,x) = 0$ for all x and $m > 0$ then implies that all N_m's with $m > 0$ remain normalized to zero and only contribute to the local density of particles, and not to the total density. This is perfectly coherent with the fact that N_0 is normalized to unity.

Define now spatial Fourier transforms by

$$\hat{f}(t,k) = \int_{\mathbb{R}^2} f(t,x)\exp(-ik.x)d^2x, \tag{21}$$

where $k.x = k_1x^1 + k_2x^2$. A direct calculation then delivers:

$$\hat{S}_1(t,k) = -k_i \int_{\mathbb{R}^2} A^i(t,k,k')\hat{N}_0(t,k-k')d^2k' \tag{22}$$

where

$$\hat{N}_0(t,k) = \exp\left(-\chi k^2 t\right) \tag{23}$$

and

$$A^i(t,k,k') = (k_j - k'_j)\hat{h}^{ij}(t,k') + \frac{1}{2}\,\eta^{ij}k'_j\eta_{kl}\hat{h}^{kl}(t,k'). \tag{24}$$

The first order density fluctuation N_1 is then obtained by solving (18) with (22) as source term, taking into account the initial condition $N_1(0,x) = 0$ for all x. One thus obtains:

$$\hat{N}_1(t,k) = I_1(t,k)\exp(-\chi k^2 t) \tag{25}$$

with

$$I_1(t,k) = \int_0^t \hat{S}_1(t',k)\exp(\chi k^2 t')dt'. \tag{26}$$

Equation (19) then gives:

$$\begin{aligned}\hat{S}_2(t,k) = -k_i &\int_{\mathbb{R}^2} A^i(t,k,k')\hat{N}_1(t,k-k')d^2k' \\ &+ \int_{\mathbb{R}^2\times\mathbb{R}^2} B^i(t,k',k'')\hat{N}_0(t,k-k')d^2k'd^2k''\end{aligned} \tag{27}$$

with

$$\begin{aligned}B^i(t,k,k') = {}&\frac{1}{2}k'_j\eta_{kl}\hat{h}^{kl}(t,k')\hat{h}^{ij}(t,k-k') \\ &- \frac{1}{4}\eta^{ij}k'_j\eta_{mk}\eta_{nl}\left(\hat{h}^{mn}(t,k')\hat{h}^{kl}(t,k-k') + \hat{h}^{kl}(t,k')\hat{h}^{mn}(t,k-k')\right).\end{aligned} \tag{28}$$

The second order density fluctuation N_2 then reads:

$$\hat{N}_2(t,k) = I_2(t,k)\exp(-\chi k^2 t) \tag{29}$$

with

$$I_2(t,k) = \int_0^t \hat{S}_2(t',k)\exp(\chi k^2 t')dt'. \tag{30}$$

4.3 How the Irregularities Influence Diffusions

4.3.1 First Order Terms

Let us now insert *Ansatz* (12) in the above expressions (22) and (25) for $\hat{S}_1$ and $\hat{N}_1$. One finds:

$$\hat{S}_1(t,k) = \sum_{nn'} \Big[A^+_{nn'}(k) \exp\big(i(\omega_{n'}t + \phi_{nn'}) - (k+k_n)^2 \chi t\big) + A^-_{nn'}(k) \exp\big(-i(\omega_{n'}t + \phi_{nn'}) - (k-k_n)^2 \chi t\big)\Big] \tag{31}$$

with

$$A^+_{nn'}(k) = -\frac{1}{2}\left[k_i(k_j + k_{nj})h^{ij}_{nn'} - \frac{1}{2}\,\eta^{ij}k_i k_{nj}\eta_{kl}h^{kl}_{nn'}\right], \tag{32}$$

$$A^-_{nn'}(k) = -\frac{1}{2}\left[k_i(k_j - k_{nj})h^{ij}_{nn'} + \frac{1}{2}\,\eta^{ij}k_i k_{nj}\eta_{kl}h^{kl}_{nn'}\right]. \tag{33}$$

This leads to:

$$\begin{aligned}\hat{N}_1(t,k) = &\sum_{(n,n')\notin\sigma^+(k)} I^+_{nn'}(k)\left[\exp\big(i\omega_{n'}t - (k+k_n)^2\chi t\big) - \exp\big(-k^2\chi t\big)\right] \\ &+ \sum_{(n,n')\notin\sigma^-(k)} I^-_{nn'}(k)\left[\exp\big(-i\omega_{n'}t - (k-k_n)^2\chi t\big) - \exp\big(-k^2\chi t\big)\right] \\ &+ t\sum_{(n,n')\in\sigma^+(k)} A^+_{nn'}(k)\exp\big(i\phi_{nn'} - k^2\chi t\big) \\ &+ t\sum_{(n,n')\in\sigma^-(k)} A^-_{nn'}(k)\exp\big(-i\phi_{nn'} - k^2\chi t\big)\end{aligned} \tag{34}$$

with

$$I^+_{nn'}(k) = \frac{A^+_{nn'}(k)\exp(i\phi_{nn'})}{i\omega_{n'} + (k^2 - (k+k_n)^2)\chi}, \tag{35}$$

$$I^-_{nn'}(k) = \frac{A^-_{nn'}(k)\exp(i\phi_{nn'})}{-i\omega_{n'} + (k^2 - (k-k_n)^2)\chi} \tag{36}$$

and $\sigma^{\pm}(k) = \{(n,n'), \pm i\omega_{n'} + (k^2 - (k \pm k_n)^2)\chi = 0\}$. Note that both sets are disjoint, unless there is an (n,n') for which $k_n = 0$ and $\omega_{n'} = 0$.

These results are best interpreted in the following way. The first order contribution $\hat{N}_1$ density is advantageously compared to the zeroth order contribution $\hat{N}_0$. One finds from (34) and (23) that:

$$\begin{aligned}\frac{\hat{N}_1(t,k)}{\hat{N}_0(t,k)} = &\sum_{(n,n')\notin\sigma^+(k)} I^+_{nn'}(k)\left[\exp\big(i\omega_{n'}t - (k_n^2 + 2k.k_n)\chi t\big) - 1\right] \\ &+ \sum_{(n,n')\notin\sigma^-(k)} I^-_{nn'}(k)\left[\exp\big(-i\omega_{n'}t - (k_n^2 - 2k.k_n)\chi t\big) - 1\right]\end{aligned}$$

$$+ t \sum_{(n,n')\in\sigma^+(k)} A^+_{nn'}(k) \exp(i\phi_{nn'})$$
$$+ t \sum_{(n,n')\in\sigma^-(k)} A^-_{nn'}(k) \exp(-i\phi_{nn'}) . \quad (37)$$

This expression characterizes how Brownian motions in the irregular metric differ, at first order, from Brownian motions on the flat Euclidean plane. The dependence on the wave vector k indicates that the influence of the irregularities varies with the spatial scale at which the diffusion is observed. Two opposite situations are particularly worth commenting upon. Take a certain k_n and consider first the value of $\hat{N}_1$ for k much smaller than k_n, say $|k| = |k_n| O(\nu)$, where ν is an infinitesimal (small parameter). Neglecting the contributions of the frequencies $\omega_{n'}$, the amplitudes $A^{\pm}_{nn'}(k)$ typically scale as $|k||k_n|$, so that the $I^{\pm}_{nn'}(k)$'s scale as $O(\nu)$. Note however that perturbations h proportional to η do not obey this typical scaling, but rather $A^{\pm}_{nn'}(k) \sim k^2$, and $I^{\pm}_{nn'}(k) \sim O(\nu^2)$. The sets $\sigma^{\pm}$ are empty and the time-dependence of $|\hat{N}_1/\hat{N}_0|$ is controlled by the real exponentials in (37), which essentially decrease as $\exp(-k_n^2 \chi t)$. The first order relative contribution $\epsilon \hat{N}_1/\hat{N}_0$ thus tends towards a quantity $L_1(k)$ which is linear in the $I^{\pm}_{nn'}(k)$; the typical relaxation time is $\tau_1 \sim 1/(\chi k_n^2)$, which is much smaller than the diffusion time $1/(\chi k^2)$ associated to the spatial scale $|k|^{-1}$. Moreover, the limit $L_1(k)$ scales as $O(\varepsilon\nu)$, except for perturbations h proportional to η, for which it scales as $O(\varepsilon\nu^2)$; $L_1(k)$ is therefore always much smaller than ε and, in particular, tends to zero with ν *i.e.* as the scale separation tends to infinity. The effect of the k_n Fourier mode on spatial scales $|k|^{-1} \gg |k_n|^{-1}$ is thus in practice negligible.

Consider now the opposite case, *i.e.* $|k|$ comparable to, or larger than $|k_n|$. Neglecting again the contribution of the frequencies $\omega_{n'}$, the amplitudes $A^{\pm}_{nn'}(k)$ then scale as k^2, and $I^{\pm}_{nn'}(k) \sim |k|/|k_n|$. Let now θ be the angle between k and k_n and suppose, to simplify the discussion, that $\cos\theta$ does not vanish. At least one of the exponentials in (37) will then be an increasing function of time provided $|k| > |k_n|/(2\cos\theta)$. Take for example $k = k_n$; the second exponential in (37) then increases with a characteristic time-scale $1/(k^2\chi)$. This means that the first order contributions of the irregularities to the density actually become comparable to unity at this scale at characteristic times $\tau_1 \sim -(\ln\varepsilon)/(k^2\chi)$; this time probably also signals the break down of the perturbative expansion in ε for the scale $k = k_n$. As for the linear terms in t appearing on the right-hand side of (37), they actually contribute to $\hat{N}_1/\hat{N}_0$ if at least one of the sets $\sigma^{\pm}(k)$ is not empty. This condition is realized if $\omega_{n'} = 0$ and $|k| = |k_n|/(2\cos\theta)$.

The conclusion of this discussion is that, at first order in the amplitude of the perturbation h, a given Fourier mode k_n of h dramatically influences diffusions on spatial scales $|k|^{-1}$ comparable or smaller to $|k_n|^{-1}$, but has a negligible influence on scales $|k|^{-1}$ much larger than $|k_n|^{-1}$. We will now show that this conclusion cannot be extended to all perturbation orders and that taking into account terms of orders higher than 1 proves that h generally influences diffusions on all scales.

4.3.2 Second Order Terms

It is straightforward to obtain from equations (27), (29), (34) and (23) explicit expressions for $\hat{S}_2(t,k)$ and $\hat{N}_2(t,k)/\hat{N}_0(t,k)$. These expressions are however extremely heavy and do not warrant reproduction in the main part of this article; an Appendix is therefore devoted to their presentation.

Consider now, for example, the contributions $D_1(t,k)$ and $D_2(t,k)$ to $\hat{N}_2(t,k)/\hat{N}_0(t,k)$ (given by (44) and (45) in the Appendix). The right-hand sides of (44) and (45) contain four exponentials of given (n, n', p, p'); these involve the wave vectors $K^{\pm}_{np} = k_n \pm k_p$. Let us for the moment ignore the factors in front of these exponentials. Let k be an arbitrary wave vector and let $\theta^{\pm}$ be the angle between k and $K^{\pm}_{np}$. Each of the conditions $2k|\cos\theta^{\pm}| > |K^{-}_{np}|$ makes one of the four exponentials an increasing function of t. At second order, the spatial scales at which diffusions are influenced by the perturbation h are thus determined, not by the k_n's, but by the combinations $K^{\pm}_{np} = k_n \pm k_p$. Indeed, quite generally, the temporal behaviour of terms of order q, $q \geq 1$, will be determined by combinations of q wave vectors k_n. For perturbations h with a rich enough spectrum, these combinations correspond to all sorts of spatial scales and, in particular, to scales much larger than those over which h itself varies. Thus, h will generally influence diffusions on all spatial scales.

Let us elaborate quantitatively on this conclusion by further exploring the behaviour of $D_1(t, k)$ and $D_2(t, k)$. Let us start with $D_1(t, k)$. Suppose for example that the moduli of all k_n's are of the same order of magnitude, conveniently denoted by K^*, but that there are some n and p for which $|K^{-}_{np}| \sim K^* O(\nu)$, where $\nu \ll 1$. The condition introduced above, which ensures that one of the exponentials involving K^{-}_{np} grows with t, then translates into $|k| > (2/\cos\theta^{-}) K^* O(\nu)$, and is realized for $|k| = K^* O(\nu)$ provided $\cos\theta^{-} \lesssim 1$. Let us check now that the factors in front of the exponentials do not tend towards zero with ν. Ignoring as before the influence of the frequencies ω_q, the quantity $I^{-}_{nn'}(k + k_p)$ (see (36)) scales as $A^{-}_{nn'}(k + k_p)/k_p^2$ *i.e.* as $k_p^2/k_p^2 = 1$. The quantity $A^{+}_{pp'}(k)$ scales as $|k||k_p|$ if h is not proportional to η, or as k^2 otherwise. Finally, the quantity $\tilde{J}^{-}_{nn'pp'}(k)$ scales as $(Q_{np}(k))^{-1} = [2k.K^{-}_{np} - (K^{-}_{np})^2]^{-1}$. The factor in front of the exponential thus scales as $|k||k_p|(Q_{np}(k))^{-1}$ for perturbations h not proportional to η, and as $k^2(Q_{np}(k))^{-1}$ otherwise. Taking into account that $|k| \sim |K_{np}|$ and putting $\cos\theta^{-} = 1$ to simplify the discussion, one finds that the factor in front of the exponentials scales as $|k_p|/|k| = O(1/\nu)$ if h is not proportional to η and as $O(1)$ otherwise. This factor therefore does not tend to zero with the separation scale parameter ν. Actually, for perturbations which are not proportional to η, this factor tends to infinity as ν tends to zero, a fact which only increases the influence of h on diffusions. The factor in front of the exponential involving K^{-}_{np} in $D_2(t, k)$ scales as $\tilde{J}^{-}_{nn'}(k) k k_p$; since $\tilde{J}^{-}_{nn'}(k)$ scales as $(1/K^{*2}) O(1/\nu^2)$, the quantity $\tilde{J}^{-}_{nn'}(k) k k_p$ scales as $O(1/\nu)$, which also tends to infinity with the scale separation.

These estimates can be used to evaluate some characteristic times. For perturbations proportional to h, the second order term $\varepsilon^2 D_1$ reaches unity after a characteristic time $\tau_2^{\eta} \sim -(2/\nu^2 K^{*2}\chi) \ln\varepsilon$; for perturbations not proportional to η, the corresponding characteristic time is $\tau_2 \sim -(2/\nu^2 K^{*2}\chi) \ln(\varepsilon/\nu^{1/2}) \ll \tau_2^{\eta}$. For all perturbations, the second order term $\varepsilon^2 D_2$ becomes of order unity after a typical time $\tilde{\tau}_2 = \tau_2$. These characteristic times are probably upper bound for the time at which the perturbation expansion ceases to be valid for scale $|k|^{-1}$.

5 Conclusion

We have investigated how metric irregularities influence Brownian motion on a differential manifold. We have performed explicit perturbative calculations for nearly flat 2D manifolds and reached the conclusion that the metric irregularities have a cumulative effect on Brownian motion; more precisely, we have found that the relative difference of the spatial Fourier components of the densities generated by a Brownian motion on the flat surface and a Brownian motion on the irregular surface grows exponentially with time on all spatial scales, including scales much larger than those characteristic of the metric perturbation; characteristic times have also been derived.

Let us conclude this article by mentioning some problems left open for further study. As stated in the introduction, many biological phenomena involve lateral diffusions on 2D interfaces. The results of this article suggest that the fluctuations of the interfaces profoundly affect these lateral diffusions; the discrepancies between real diffusions on irregular interfaces and idealized diffusions on highly regular surfaces are therefore probably observable and the biological consequences of these discrepancies should be carefully studied. On the theoretical side, one should envisage a non perturbative treatment of at least some of the problems studied in this article; this is probably best achieved through numerical simulations; a first step would be to confirm numerically, at least for 2D diffusions, the characteristic time estimates we have derived here. Finally, the case of relativistic diffusions in fluctuating space-times is certainly worth investigating, notably in a cosmological context.

Appendix

Equations (27), (34), (23) lead to:

$$\hat{S}_2(t,k) = C_1(t,k) + C_2(t,k) + C_3(t,k) \tag{38}$$

where

$$\begin{aligned}
C_1(t,k) = &\sum_{\substack{(n,n')\notin\sigma^+(k)\\ pp'}} I^+_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right)\\
&\times\left[\exp\left(i(\omega_{n'}+\omega_{p'})t-\left(k+k_n+k_p\right)^2\chi t\right)-\exp\left(i\omega_{p'}t-\left(k+k_p\right)^2\chi t\right)\right]\\
&+\sum_{\substack{(n,n')\notin\sigma^+(k)\\ pp'}} I^+_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right)\\
&\times\left[\exp\left(i(\omega_{n'}-\omega_{p'})t-\left(k+k_n-k_p\right)^2\chi t\right)-\exp\left(-i\omega_{p'}t-\left(k-k_p\right)^2\chi t\right)\right]\\
&+\sum_{\substack{(n,n')\notin\sigma^+(k)\\ pp'}} I^-_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right)\\
&\times\left[\exp\left(i(-\omega_{n'}+\omega_{p'})t-\left(k-k_n+k_p\right)^2\chi t\right)-\exp\left(i\omega_{p'}t-\left(k+k_p\right)^2\chi t\right)\right]\\
&+\sum_{\substack{(n,n')\notin\sigma^+(k)\\ pp'}} I^-_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right)\\
&\times\left[\exp\left(-i(\omega_{n'}+\omega_{p'})t-\left(k-k_n-k_p\right)^2\chi t\right)-\exp\left(-i\omega_{p'}t-\left(k-k_p\right)^2\chi t\right)\right],
\end{aligned} \tag{39}$$

$$\begin{aligned}
C_2(t,k) = &\sum_{nn'pp'} k_i k_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(i(\phi_{nn'}+\phi_{pp'})\right)\exp\left(i(\omega_{n'}+\omega_{p'})t-(k+k_n+k_p)^2\chi t\right)\\
&+\sum_{nn'pp'} k_i k_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(i(\phi_{nn'}-\phi_{pp'})\right)\exp\left(i(\omega_{n'}-\omega_{p'})t-(k+k_n-k_p)^2\chi t\right)
\end{aligned}$$

$$- \sum_{nn'pp'} k_i k_{nj} \tilde{B}^{ij}_{nn'pp'} \exp\left(i(-\phi_{nn'} + \phi_{pp'})\right)$$
$$\times \exp\left(i(-\omega_{n'} + \omega_{p'})t - (k - k_n + k_p)^2 \chi t\right)$$
$$- \sum_{nn'pp'} k_i k_{nj} \tilde{B}^{ij}_{nn'pp'} \exp\left(-i(\phi_{nn'} + \phi_{pp'})\right)$$
$$\times \exp\left(i - (\omega_{n'} + \omega_{p'})t - (k - k_n - k_p)^2 \chi t\right), \tag{40}$$

with

$$\tilde{B}^{ij}_{nn'pp'} = \frac{1}{8}\left[\eta_{kl} h^{kl}_{nn'} h^{ij}_{pp'} - \frac{1}{2}\eta^{ij}\eta_{qk}\eta_{rl}\left(h^{qr}_{nn'} h^{kl}_{pp'} + h^{qr}_{pp'} h^{kl}_{nn'}\right)\right] \tag{41}$$

and

$$C_3(t,k) = t \sum_{\substack{(n,n')\in\sigma^+(k)\\ pp'}} A^+_{nn'}(k+k_p) A^+_{pp'}(k) \exp\left(i(\phi_{nn'} + \phi_{pp'})\right) \exp\left(i\omega_{p'} t - \left(k + k_p\right)^2 \chi t\right)$$
$$+ t \sum_{\substack{(n,n')\in\sigma^+(k)\\ pp'}} A^+_{nn'}(k-k_p) A^-_{pp'}(k)$$
$$\times \exp\left(i(\phi_{nn'} - \phi_{pp'})\right) \exp\left(-i\omega_{p'} t - \left(k - k_p\right)^2 \chi t\right)$$
$$+ t \sum_{\substack{(n,n')\in\sigma^-(k)\\ pp'}} A^-_{nn'}(k+k_p) A^+_{pp'}(k)$$
$$\times \exp\left(i(-\phi_{nn'} + \phi_{pp'})\right) \exp\left(i\omega_{p'} t - \left(k + k_p\right)^2 \chi t\right)$$
$$+ t \sum_{\substack{(n,n')\in\sigma^-(k)\\ pp'}} A^-_{nn'}(k-k_p) A^-_{pp'}(k)$$
$$\times \exp\left(-i(\phi_{nn'} + \phi_{pp'})\right) \exp\left(-i\omega_{p'} t - \left(k - k_p\right)^2 \chi t\right). \tag{42}$$

We get, from (29), (38) and (23):

$$\frac{\hat{N}_2(t,k)}{\hat{N}_0(t,k)} = \sum_{i=1}^{8} D_i(t,k) \tag{43}$$

where

$$D_1(t,k) = \sum_{\substack{(n,n')\notin\sigma^+(k)\\ (n,n',p,p')\notin\tau^+(k)}} I^+_{nn'}(k+k_p) A^+_{pp'}(k) J^+_{nn'pp'}(k)$$
$$\times \left[\exp\left(i(\omega_{n'} + \omega_{p'})t - \left((k_n + k_p)^2 + 2k.(k_n + k_p)\right)\chi t\right) - 1\right]$$
$$+ \sum_{\substack{(n,n')\notin\sigma^+(k)\\ (n,n',p,p')\notin\tilde{\tau}^+(k)}} I^+_{nn'}(k-k_p) A^-_{pp'}(k) \tilde{J}^+_{nn'pp'}(k)$$
$$\times \left[\exp\left(i(\omega_{n'} - \omega_{p'})t - \left((k_n - k_p)^2 + 2k.(k_n - k_p)\right)\chi t\right) - 1\right]$$

$$
\begin{aligned}
&+ \sum_{\substack{(n,n')\notin\sigma^-(k)\\(n,n',p,p')\notin\tilde{\tau}^-(k)}} I^-_{nn'}(k+k_p)A^+_{pp'}(k)\tilde{J}^-_{nn'pp'}(k)\\
&\times\left[\exp\left(i(-\omega_{n'}+\omega_{p'})t-\left((k_n-k_p)^2-2k.(k_n-k_p)\right)\chi t\right)-1\right]\\
&+ \sum_{\substack{(n,n')\notin\sigma^-(k)\\(n,n',p,p')\notin\tau^-(k)}} I^-_{nn'}(k-k_p)A^-_{pp'}(k)J^-_{nn'pp'}(k)\\
&\times\left[\exp\left(-i(\omega_{n'}+\omega_{p'})t-\left((k_n+k_p)^2-2k.(k_n+k_p)\right)\chi t\right)-1\right], \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
D_2(t,k) = &\sum_{(n,n',p,p')\notin\tau^+(k)} J^+_{nn'pp'}(k)k_ik_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(i\phi_{nn'}\right)\\
&\times\left[\exp\left(i(\omega_{n'}+\omega_{p'})t-\left((k_n+k_p)^2+2k.(k_n+k_p)\right)\chi t\right)-1\right]\\
&+\sum_{(n,n',p,p')\notin\tilde{\tau}^+(k)} \tilde{J}^+_{nn'pp'}(k)k_ik_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(i\phi_{nn'}\right)\\
&\times\left[\exp\left(i(\omega_{n'}-\omega_{p'})t-\left((k_n-k_p)^2+2k.(k_n-k_p)\right)\chi t\right)-1\right]\\
&-\sum_{(n,n',p,p')\notin\tilde{\tau}^-(k)} \tilde{J}^-_{nn'pp'}(k)k_ik_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(-i\phi_{nn'}\right)\\
&\times\left[\exp\left(i(-\omega_{n'}+\omega_{p'})t-\left((k_n-k_p)^2-2k.(k_n-k_p)\right)\chi t\right)-1\right]\\
&-\sum_{(n,n',p,p')\notin\tau^-(k)} J^-_{nn'pp'}(k)k_ik_{nj}\tilde{B}^{ij}_{nn'pp'}\exp\left(-i\phi_{nn'}\right)\\
&\times\left[\exp\left(-i(\omega_{n'}+\omega_{p'})t-\left((k_n+k_p)^2-2k.(k_n+k_p)\right)\chi t\right)-1\right], \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
D_3(t,k) = &-\sum_{\substack{(n,n')\notin\sigma^+(k)\\(p,p')\notin\sigma^+(k)}} I^+_{nn'}(k+k_p)I^+_{pp'}(k)\left[\exp\left(i\omega_{p'}t-\left(k_p^2+2k.k_p\right)\chi t\right)-1\right]\\
&-\sum_{\substack{(n,n')\notin\sigma^+(k)\\(p,p')\notin\sigma^-(k)}} I^+_{nn'}(k-k_p)I^-_{pp'}(k)\left[\exp\left(-i\omega_{p'}t-\left(k_p^2-2k.k_p\right)\chi t\right)-1\right]\\
&-\sum_{\substack{(n,n')\notin\sigma^-(k)\\(p,p')\notin\sigma^+(k)}} I^-_{nn'}(k+k_p)I^+_{pp'}(k)\left[\exp\left(i\omega_{p'}t-\left(k_p^2+2k.k_p\right)\chi t\right)-1\right]\\
&-\sum_{\substack{(n,n')\notin\sigma^-(k)\\(p,p')\notin\sigma^-(k)}} I^-_{nn'}(k-k_p)I^-_{pp'}(k)\left[\exp\left(-i\omega_{p'}t-\left(k_p^2-2k.k_p\right)\chi t\right)-1\right], \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
D_4(t,k) = t &\sum_{\substack{(n,n')\notin\sigma^+(k)\\(n,n',p,p')\in\tau^+(k)}} I^+_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right)\\
&+t\sum_{\substack{(n,n')\notin\sigma^+(k)\\(n,n',p,p')\in\tilde{\tau}^+(k)}} I^+_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right)
\end{aligned}
$$

$$+ t \sum_{\substack{(n,n')\notin\sigma^-(k) \\ (n,n',p,p')\in\tilde{\tau}^-(k)}} I^-_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right)$$

$$+ t \sum_{\substack{(n,n')\notin\sigma^-(k) \\ (n,n',p,p')\in\tau^-(k)}} I^-_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right), \tag{47}$$

$$\begin{aligned} D_5(t,k) = &-t \sum_{\substack{(n,n')\notin\sigma^+(k) \\ (p,p')\in\sigma^+(k)}} I^+_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right) \\ &- t \sum_{\substack{(n,n')\notin\sigma^+(k) \\ (p,p')\in\sigma^-(k)}} I^+_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right) \\ &- t \sum_{\substack{(n,n')\notin\sigma^-(k) \\ (p,p')\in\sigma^+(k)}} I^-_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i\phi_{pp'}\right) \\ &- t \sum_{\substack{(n,n')\notin\sigma^-(k) \\ (p,p')\in\sigma^-(k)}} I^-_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i\phi_{pp'}\right), \end{aligned} \tag{48}$$

$$\begin{aligned} D_6(t,k) = &\, t \sum_{(n,n',p,p')\in\tau^+(k)} k_i k_{nj}\,\tilde{B}^{ij}_{nn'pp'}\exp\left(i(\phi_{nn'}+\phi_{pp'})\right) \\ &+ t \sum_{(n,n',p,p')\in\tilde{\tau}^+(k)} k_i k_{nj}\,\tilde{B}^{ij}_{nn'pp'}\exp\left(i(\phi_{nn'}-\phi_{pp'})\right) \\ &- t \sum_{(n,n',p,p')\in\tilde{\tau}^-(k)} k_i k_{nj}\,\tilde{B}^{ij}_{nn'pp'}\exp\left(i(-\phi_{nn'}+\phi_{pp'})\right) \\ &- t \sum_{(n,n',p,p')\in\tau^-(k)} k_i k_{nj}\,\tilde{B}^{ij}_{nn'pp'}\exp\left(-i(\phi_{nn'}+\phi_{pp'})\right), \end{aligned} \tag{49}$$

$$\begin{aligned} D_7(t,k) = &\, t \sum_{\substack{(n,n')\in\sigma^+(k) \\ (p,p')\notin\sigma^+(k)}} A^+_{nn'}(k+k_p)I^+_{pp'}(k)\exp\left(i\phi_{nn'}\right)\left[\exp\left(i\omega_{p'}t-\left(k_p^2+2k.k_p\right)\chi t\right)-1\right] \\ &+ t \sum_{\substack{(n,n')\in\sigma^+(k) \\ (p,p')\notin\sigma^-(k)}} A^+_{nn'}(k-k_p)I^-_{pp'}(k)\exp\left(i\phi_{nn'}\right) \\ &\times\left[\exp\left(-i\omega_{p'}t-\left(k_p^2-2k.k_p\right)\chi t\right)-1\right] \\ &+ t \sum_{\substack{(n,n')\in\sigma^-(k) \\ (p,p')\notin\sigma^+(k)}} A^-_{nn'}(k+k_p)I^+_{pp'}(k)\exp\left(-i\phi_{nn'}\right) \\ &\times\left[\exp\left(i\omega_{p'}t-\left(k_p^2+2k.k_p\right)\chi t\right)-1\right] \\ &+ t \sum_{\substack{(n,n')\in\sigma^-(k) \\ (p,p')\notin\sigma^-(k)}} A^-_{nn'}(k-k_p)I^-_{pp'}(k)\exp\left(-i\phi_{nn'}\right) \\ &\times\left[\exp\left(-i\omega_{p'}t-(k-k_p)^2\chi t\right)-1\right], \end{aligned} \tag{50}$$

$$\begin{aligned} D_8(t,k) = t^2 & \sum_{\substack{(n,n')\in\sigma^+(k)\\(p,p')\in\sigma^+(k)}} A^+_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i(\phi_{nn'}+\phi_{pp'})\right) \\ & + t^2 \sum_{\substack{(n,n')\in\sigma^+(k)\\(p,p')\in\sigma^-(k)}} A^+_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(i(\phi_{nn'}-\phi_{pp'})\right) \\ & + t^2 \sum_{\substack{(n,n')\in\sigma^-(k)\\(p,p')\in\sigma^+(k)}} A^-_{nn'}(k+k_p)A^+_{pp'}(k)\exp\left(i(-\phi_{nn'}+\phi_{pp'})\right) \\ & + t^2 \sum_{\substack{(n,n')\in\sigma^-(k)\\(p,p')\in\sigma^-(k)}} A^-_{nn'}(k-k_p)A^-_{pp'}(k)\exp\left(-i(\phi_{nn'}+\phi_{pp'})\right), \end{aligned} \tag{51}$$

with

$$J^+_{nn'pp'}(k) = \frac{\exp(i\phi_{pp'})}{i(\omega_{n'}+\omega_{p'})+(k^2-(k+k_n+k_p)^2)\chi}, \tag{52}$$

$$\tilde{J}^+_{nn'pp'}(k) = \frac{\exp(-i\phi_{pp'})}{i(\omega_{n'}-\omega_{p'})+(k^2-(k+k_n-k_p)^2)\chi}, \tag{53}$$

$$\tilde{J}^-_{nn'pp'}(k) = \frac{\exp(i\phi_{pp'})}{i(-\omega_{n'}+\omega_{p'})+(k^2-(k-k_n+k_p)^2)\chi}, \tag{54}$$

$$J^-_{nn'pp'}(k) = \frac{\exp(-i\phi_{pp'})}{-i(\omega_{n'}+\omega_{p'})+(k^2-(k-k_n-k_p)^2)\chi} \tag{55}$$

and

$$\tau^\pm(k) = \pm i(\omega_{n'}+\omega_{p'})+(k^2-(k\pm k_n\pm k_p)^2)\chi, \tag{56}$$

$$\tilde{\tau}^+(k) = i(\omega_{n'}-\omega_{p'})+(k^2-(k+k_n-k_p)^2)\chi, \tag{57}$$

$$\tilde{\tau}^-(k) = i(-\omega_{n'}+\omega_{p'})+(k^2-(k-k_n+k_p)^2)\chi. \tag{58}$$

References

1. Abarbanel, S., Ditkowski, A.: Asymptotically stable fourth-order accurate schemes for the diffusion equation on complex shapes. J. Comput. Phys. **133**, 279 (1996)
2. Allen, L.J.S.: An Introduction to Stochastic Processes with Applications to Biology. Prentice Hall, Englewood Cliffs (2003)
3. Braga, J., Desterro, J.M.P., Carmo-Fonseca, M.: Mol. Biol. Cell **15**, 4749–4760 (2004)
4. Brünger, A., Peters, R., Schulten, K.: Continuous fluorescence microphotolysis to observe lateral diffusion in membranes: theoretical methods and applications. J. Chem. Phys. **82**, 2147 (1984)
5. Chevalier, C., Debbasch, F.: Fluctuation-dissipation theorems in an expanding Universe. J. Math. Phys. **48**, 023304 (2007)
6. Christensen, M.: How to simulate anisotropic diffusion processes on curved surfaces. J. Comput. Phys. **201**, 421–435 (2004)
7. Dubrovin, B.A., Novikov, S.P., Fomenko, A.T.: Modern Geometry—Methods and Applications. Springer, New York (1984)
8. Edelstein-Keshet, L.: Mathematical Models in Biology. Classics in Applied Mathematics, vol. 46. SIAM, Philadelphia (2005)

9. Emery, M.: Stochastic Calculus in Manifolds. Springer, New York (1989)
10. Goel, N.S., Richter-Dyn, N.: Stochastic Models in Biology. Blackburn, Caldwell (2004)
11. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes, 2nd edn. North-Holland Mathematical Library. North-Holland, Amsterdam (1989)
12. Itô, K.: On stochastic differential equations on a differentiable manifold. I. Nagoya Math. J. **1**, 35–47 (1950)
13. Itô, K.: On stochastic differential equations on a differentiable manifold. II. Mem. Coll. Sci. Univ. Kyoto **28**, 82–85 (1953)
14. McKean, H.P.: Stochastic Integrals. Academic Press, New York (1969)
15. Mitra, D., Wang, Q.: Stochastic traffic engineering for demand uncertainty and risk-aware network revenue management. IEEE/ACM Trans. Netw. **13**(2), 221–233 (2005)
16. Murray, J.D.: Mathematical Biology I: An Introduction, 3rd edn. Interdisciplinary Applied Mathematics, Mathematical Biology. Springer, New York (2002)
17. Nehls, S., et al.: Dynamics and retention of misfolded proteins in native ER membranes. Nat. Cell Biol. **2**, 288–295 (2000)
18. Øksendal, B.: Stochastic Differential Equations, 5th edn. Springer, Berlin (1998)
19. Sbalzarini, I.F., Hayer, A., Helenius, A., Koumoutsakos, P.: Simulations of (an)isotropic diffusion on curved biological interfaces. Biophys. J. **90**(3), 878–885 (2006)
20. Schreckenberg, M., Schadschneider, A., Nagel, K., Ito, N.: Discrete stochastic models for traffic flow. Phys. Rev. E **51**(4), 2939–2949 (1995)
21. Shreve, S.E.: Stochastic Calculus for Finance I: The Binomial Asset Pricing Model. Springer Finance. Springer, New York (2004)
22. Shreve, S.E.: Stochastic Calculus for Finance II: Continuous-Time Models. Springer Finance. Springer, New York (2004)
23. van Kampen, N.G.: Stochastic Processes in Physics and Chemistry. North-Holland, Amsterdam (1992)